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Equilibrium Points for Optimal Investment with Vintage Capital

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EQUILIBRIUM POINTS FOR OPTIMAL INVESTMENT WITH VINTAGE CAPITAL

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Abstract. The paper concerns the study of equilibrium points, namely the stationary solutions to the closed loop equation, of an infinite dimensional and infinite horizon boundary control problem for linear partial differential equations. Sufficient conditions for existence of equilibrium points in the general case are given and later applied to the economic problem of optimal investment with vintage capital. Explicit computation of equilibria for the economic problem in some relevant examples is also provided. Indeed the challenging issue here is showing that a theoretical machinery, such as optimal control in infinite dimension, may be effectively used to compute solutions explicitly and easily, and that the same computation may be straightforwardly repeated in examples yielding the same abstract structure. No stability result is instead provided: the work here contained has to be considered as a first step in the direction of studying the behavior of optimal controls and trajectories in the long run.

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1. Introduction

The paper concerns the study of equilibrium points of an infinite dimensional and infinite horizon boundary control problem for linear partial differential equations. More precisely, we take into account a state equation of type

$$\begin{cases} y'(\tau) = A_0 y(\tau) + Bu(\tau), & \tau \in [t, +\infty) \\ y(t) = x \in H, \end{cases} \tag{1.1}$$

where H is the state space, $y : [t, +\infty) \to H$ is the trajectory, U is the control space and $u : [t, +\infty) \to U$ is the control, $A_0 : D(A_0) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $\{e^{\tau A_0}\}_{\tau \geq 0}$ on H, and the control operator B is linear and *unbounded*, say $B : U \to [D(A_0^*)]'$. Besides, we consider a cost functional given by

$$J_\infty(t, x, u) = \int_t^{+\infty} e^{-\lambda\tau} \left[g_0\left(y(\tau)\right) + h_0\left(u(\tau)\right) \right] d\tau \tag{1.2}$$

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where the functions g_0 and h_0 are convex functions as better specified later.

More precisely, by *equilibrium points* we mean stationary solutions to the closed loop equation associated by Dynamic Programming to (1.1) that is

$$y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}B(h_0^*)'(-B^*\Psi'(y(s)))d\sigma, \quad \tau \in [t, +\infty[, \tag{1.3}$$

where h_0^* is the convex conjugate of h_0, Ψ is the value function of the optimal control problem for initial time $t=0$, more precisely

$$\Psi(x) = Z_\infty(0,x) = \inf_{u \in L^p_\lambda(0,+\infty;U)} J_\infty(0,x,u),$$

and

$$G(x) = (h_0^*)'(-B^*\Psi'(x)),$$

is the unique optimal feedback map, as shown in [30, 31]. Indeed the problem of minimizing $J_\infty(t,x,u)$ with respect to u over the Banach space

$$L^p_\lambda(t,+\infty;U) = \{u : [t,+\infty) \to U \ : \ \tau \mapsto u(\tau)e^{-\frac{\lambda}{p}\tau} \in L^p(t,+\infty;U)\}, \ p \geq 2,$$

was studied paper by Faggian and Gozzi in [30], and by Faggian in [31] by means of Dynamic Programming methods, deriving:

- existence and uniqueness for the associated Hamilton-Jacobi-Bellman (briefly, HJB) equation
- a feedback formula for optimal controls in terms of the spatial gradient of the value function,
- Pontryagin Maximum Principle.

All of these results are recalled in Section 2.
As a first result here, we give sufficient conditions for existence of equilibrium points in the general case, we apply such results to the problem optimal investment with vintage capital described in Section 3 (cfr. Section 4). Nevertheless the most interesting result of the paper is the explicit computation of equilibria for the economic problem in some relevant examples (Section 5). Indeed the challenging issue here is showing that a theoretical machinery such as optimal control in infinite dimension may be effectively used to *compute* solutions explicitly and easily, and that the same computation may be straightforwardly repeated in examples yielding the same abstract structure.
No stability result is instead provided. Under this respect, the work here contained has to be considered as a first step in the direction of studying the behavior of optimal controls and trajectories in the long run.

1.1. **Bibliographical notes.** It is well known that control problems with unbounded control operator B arise when we rephrase into abstract terms some boundary control problem for PDEs (or, more generally, problems with control on a subdomain). Indeed, we motivate our framework with the application to the economic problem of optimal investment with vintage capital in the framework by Barucci and Gozzi [12] [13] that we describe in detail in Section 3. Similar problems with unbounded control operator have been discussed in a series of papers by this author and others. The unconstrained case has been studied both in the case of finite and infinite horizon [26, 27, 30] while [29] contains the finite horizon case with constrained controls. The (finite horizon) case with both boundary control and state constraints is treated in [28]. The case of infinite horizon (without constraints) has been treated in [30] and [31].

Some further references on *boundary control* in infinite dimension follow. We recall that such problems have been studied in the framework of classical/strong solutions and in that of viscosity solutions. Regarding Dynamic Programming in the classical/strong framework, the available results mainly regard the case of linear systems and quadratic costs (where HJB reduces to the operator Riccati equation). The reader is then referred *e.g.* to the book by Lasiecka and Triggiani [42], to the book by Bensoussan, Da Prato, Delfour and Mitter [14], and, for the case of nonautonomous systems, to the papers by Acquistapace, Flandoli and Terreni [2, 3, 4, 5]. For the case of a linear system and a general convex cost, we mention the papers by this author [24, 25, 26, 27]. On Pontryagin maximum principle for boundary control problems we mention again the book by Barbu and Precupanu (Chapter 4 in [11]).

For viscosity solutions and HJB equations in infinite dimension we mention the series of papers by Crandall and Lions [18] where also some boundary control problem arises. Moreover, for boundary control we mention Gozzi, Cannarsa and Soner [17] and the paper by Cannarsa and Tessitore [19] on existence and uniqueness of viscosity solutions of HJB. We note also that a verification theorem in the case of viscosity solutions has been proved in some finite dimensional case in the book by Yong and Zhou [44]. We finally mention the paper by Fabbri [23] where the author derives an existence and uniqueness result for the viscosity solution of HJB associated to optimal investment with vintage capital (with infinite horizon and without constraints), that is the application of Section 3 of the present paper, obtaining the results by making use of the specific properties of the state equation, while no result is there provided for the general problem.

We mention also some fundamental papers and books on the case of *distributed control* in the classical/strong framework such as the works by Barbu and Da Prato [7, 8, 9] for some linear convex problems, to Di Blasio [20, 21] for the case of constrained control, to Cannarsa and Di Blasio [16] for the case of state constraints, to Barbu, Da Prato and Popa [10] and to Gozzi [36, 37, 38] for semilinear systems.

Regarding applications, on control on a subdomain (boundary or point control) we refer the reader to the many examples contained in the books by Lasiecka and Triggiani [42], and by Bensoussan *et al* [14]. Moreover, for economic models with vintage capital the reader may see the papers by Barucci and Gozzi [12], [13], the papers by Feichtinger, Hartl, Kort, Veliov et al. [32, 33, 34, 35], and for population dynamic the book by Iannelli [41], the paper by Aniţa, Iannelli, Kim and Park [1], and the papers by Almeder, Caulkins, Feichtinger, Tragler, and Veliov [6] and references therein.

2. Preliminaries

We here recall all the relevant results that are needed in the sequel. The reader may find the proofs of all statements in [30, 31]. According to the notation there contained, if X and Y are Banach spaces, we denote by $|\cdot|_X$ the norm on X, by $|\cdot|$ the euclidean norm in $\mathbb{R}$, and we set

$$Lip(X;Y) = \{f : X \to Y \ : \ [f]_L := \sup_{x,y\in X,\ x\neq y} \frac{|f(x)-f(y)|_Y}{|x-y|_X} < +\infty\}$$
$$C^1_{Lip}(X) := \{f \in C^1(X) \ : \ [f']_L < +\infty\}$$
$$\mathcal{B}_r(X,Y) := \{f : X \to Y \ : \ |f|_{\mathcal{B}_r} := \sup_{x\in X} \frac{|f(x)|_Y}{1+|x|_X^r} < +\infty\}, \quad \mathcal{B}_r(X) := \mathcal{B}_r(X,\mathbb{R}).$$

Moreover we set

$$\Sigma_0(X) := \{w \in \mathcal{B}_2(X) \ : \ w \text{ is convex, } w \in C^1_{Lip}(X)\}$$

and, for $T > 0$

$$\begin{aligned}\mathcal{Y}([0,T]\times X) = \{w : [0,T]\times X \to \mathbb{R} \ : \ & w \in C([0,T],\mathcal{B}_2(X)),\\ & w(t,\cdot) \in \Sigma_0(X),\ \forall t \in [0,T],\ \ w_x \in C([0,T],\mathcal{B}_1(X))\}\end{aligned}$$

All the spatial derivatives above have to be intended as Frechét differentials.

Then we consider two Hilbert spaces V, V', being dual spaces, which we do not identify for reasons which are recalled in Remark 2.2 and we denote the duality pairing by $\langle\cdot,\cdot\rangle$. We set V' as the state space of the problem, and denote with U the control space, being U another Hilbert space. The state space is V' and the control space is U. For any fixed x in V' and $t > 0$ and $\tau \geq t$, the solution to the state equation in V' is given by variation of constant formula by

$$y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}Bu(\sigma)d\sigma, \quad \tau \in [t,+\infty[, \tag{2.1}$$

while the target functional is of type

$$J_\infty(t,x,u) := \int_t^{+\infty} e^{-\lambda\tau}[g_0(y(\tau)) + h_0(u(\tau))]d\tau. \tag{2.2}$$

We assume the following hypotheses hold:

Assumptions 2.1.
1. $A : D(A) \subset V' \to V'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{\tau A}\}_{\tau\geq 0}$ on V';
2. $B \in L(U,V')$;
3. there exists $\omega \geq 0$ such that $|e^{\tau A}x|_{V'} \leq e^{\omega\tau}|x|_{V'}$, $\forall\tau \geq 0$;
4. $g_0, \phi_0 \in \Sigma_0(V')$
5. h_0 is convex, lower semi–continuous, $\partial_u h_0$ is injective.
6. $h_0^*(0) = 0$, $h_0^* \in \Sigma_0(V)$;

7. $\exists a > 0$, $\exists b \in \mathbb{R}$, $\exists p \geq 2$: $h_0(u) \geq a|u|_U^p + b$, $\forall u \in U$;
   Moreover, either

8.a $p > 2$, $\lambda > 2\omega$.
   or

8.b $\lambda > \omega$, and $g_0, \phi_0 \in \mathcal{B}_1(V')$.

Remark 2.2. We do not identify V and V' for in the applications the problem is naturally set in a Hilbert space H, such that $V \subset H \equiv H' \subset V'$ (with all bounded inclusions). Indeed, in order to avoid the discontinuities due to the presence of B, as they appear in (1.1)(1.2), we work in the extended state space V' related to H in the following way: V is the Hilbert space $D(A_0^*)$ endowed with the scalar product $(v|w)_V := (v|w)_H + (A_0^*v|A_0^*w)_H$, V' is the dual space of V endowed with the operator norm. Then assume that $B \in L(U, V')$, and extend the semigroup $\{e^{tA_0}\}_{t\geq 0}$ on H to a semigroup $\{e^{tA}\}_{t\geq 0}$ on the space V', having infinitesimal generator A, a proper extension of A_0. The reader is referred to [27] for a detailed treatment. The coefficient ω could be any real number, but is assumed positive in order to avoid double proofs for positive and negative signs. □

Remark 2.3. Note that the functions g and φ arising from applications usually appear to be defined and C^1 on H, not on the larger space V'. Then, we here need to *assume* that they can be extended to *C^1-regular functions on V'* - which is a non trivial issue. We refer the reader to [26], [27] and [30] for a thorough discussion on this issue. □

The functional $J_\infty(t; x, u)$ has to be minimized with respect to u over the set of admissible controls

$$L_\lambda^p(t, +\infty; U) = \{u : [t, +\infty) \to U \ ; \ \tau \mapsto u(\tau)e^{-\frac{\lambda\tau}{p}} \in L^p(t, +\infty; U)\}, \tag{2.3}$$

which is Banach space with the norm

$$\|u\|_{L_\lambda^p(t,+\infty;U)} = \int_t^{+\infty} |u(\tau)|_U^p e^{-\lambda\tau} d\tau = \|e^{-\frac{\lambda(\cdot)}{p}} u\|_{L^p(t,+\infty;U)}.$$

The value function is then defined as

$$Z_\infty(t, x) = \inf_{u \in L_\lambda^p(t,+\infty;U)} J_\infty(t, x, u).$$

As it is easy to check that

$$Z_\infty(t, x) = e^{-\lambda t} Z_\infty(0, x)$$

one may associate to the problem the following stationary HJB equation

$$-\lambda\psi(x) + \langle\psi'(x), Ax\rangle - h_0^*(-B^*\psi'(x)) + g(x) = 0, \tag{2.4}$$

whose candidate solution is the function $Z_\infty(0, \cdot)$.

We will use the following definition of solution for equation (2.4).

Definition 2.4. *A function ψ is a classical solution of the stationary HJB equation (2.4) if it belongs to $\Sigma_0(V')$ and satisfies (2.4) for every $x \in D(A)$.*

Theorem 2.5. *Let Assumptions 2.1 hold. Then there exists a unique classical solution* Ψ *to* (2.4) *and it is given by the value function of the optimal control problem, that is*

$$\Psi(x) = Z_\infty(0,x) = \inf_{u \in L^p_\lambda(0,+\infty;U)} J_\infty(0,x,u).$$

Once we have established that Ψ is the classical solution to the stationary HJB equation, and that it is differentiable, we can build optimal feedbacks and prove the following theorem.

Theorem 2.6. *Let Assumptions 2.1 hold. Let* $t \geq 0$ *and* $x \in V'$ *be fixed. Then there exists a unique optimal pair* (u^*, y^*). *The optimal state* y^* *is the unique solution of the Closed Loop Equation*

$$y(\tau) = e^{(\tau - t)A}x + \int_t^\tau e^{(\tau-\sigma)A} B (h_0^*)'(-B^*\Psi'(y(s)))d\sigma, \quad \tau \in [t, +\infty[. \tag{2.5}$$

while the optimal control u^* *is given by the feedback formula*

$$u^*(s) = (h_0^*)'(-B^*\Psi'(y^*(s))).$$

Let $x \in V'$ and $t \geq 0$ be fixed, and consider the dual system associated to (1.1), that is

$$\pi'(\tau) = (\lambda - A_0^*)\pi(\tau) - g_0'(y(\tau)), \qquad \tau \in [t, +\infty) \tag{2.6}$$

where $\pi : [t, +\infty) \to V$ (the dual variable, or co-state of the system) is the unknown, and $y = y(\cdot; t, x, u)$ is the trajectory starting at x at time t and driven by control u, given by (2.1). We assume such equation is also subject to the following transversality condition

$$\lim_{T \to +\infty} \pi(T) = 0. \tag{2.7}$$

We denote any solution of (2.6)(2.7) also by $\pi(\cdot; t, x, u)$ or by $\pi(\cdot; t, x)$ to remark its dependence on the data.

Definition 2.7. *Let Assumptions 2.1 [1-7] be satisfied. We define the mild solution of (2.6)(2.7) as the function* $\pi : [t, +\infty) \to V$ *given by*

$$\pi(\tau) = \int_\tau^{+\infty} e^{(A_0^* - \lambda)(\sigma - \tau)} g_0'(y(\sigma))d\sigma. \tag{2.8}$$

In the sequel we show that such definition is natural.

Lemma 2.8. *Let Assumption 2.1 [1-7] be satisfied, and assume* $p \geq 2$ *and* $\lambda > 2\omega$. *Then* π *given by (2.8) is well defined and belongs to* $C^0(t, +\infty; V)$.
Moreover:
(i) if $p > 2$ *then* $\pi \in L^q_\lambda(t, +\infty; V)$;
(ii) if $p = 2$ *then* $\pi \in L^2_{\lambda+\varepsilon}(t, +\infty; V) \cap L^2(t, T; V)$, $\forall\ T < +\infty,\ \varepsilon > 0$.

Theorem 2.9. *If* $\pi \in W^{1,1}(t, +\infty; V)$ *satisfies* (2.6) *almost everywhere in* $[t, +\infty)$ *and* (2.7) *then* π *is given by (2.8), that is* π *is the mild solution of* (2.6)(2.7).

Remark 2.10. Assume $p \geq 2$, $\lambda > 2\omega$. Then:

- $\lambda \le \omega p$ implies $y \in L^r(t,+\infty;V')$ for all $r < \frac{\lambda}{p}$, and $y \in L^{\frac{\lambda}{p}}(t,T;V')$ for all $T < +\infty$;
- $\lambda > \omega p$ implies $y \in L^r(t,+\infty;V')$ for all $r < p$, and $y \in L^p(t,T;V')$ for all $T < +\infty$.

Definition 2.11. *Let Assumption 2.1 [1-7] be satisfied, and assume $p \ge 2$ and $\lambda > 2\omega$. Let also $T > t$ be either finite or $+\infty$. We say that a given pair $(u,y) \in L^p_\lambda(t,T;U) \times L^1_{loc}(t,T;V')$ is* extremal *if and only if there exists a function $\pi \in L^q_\lambda(t,T;V)$ satisfying in mild sense, along with u and y, the following set of equations*

$$y'(\tau) = Ay(\tau) + Bu(\tau),\ \tau \in [t,T);\quad y(t) = x$$

$$\pi'(\tau) = (\lambda - A_0^*)\pi(\tau) - g_0'(y(\tau)),\ \tau \in [t,T);$$

$$\lim_{s\to+\infty} \pi(s) = 0, when\ T = +\infty;\quad \pi(T) = 0,\ when\ T < +\infty$$

$$-B^*\pi(\tau) \in \partial h_0(u(\tau)),\ for\ a.a.\ \tau \in [t,T). \tag{2.9}$$

Theorem 2.12. (Maximum Principle). *Let Assumptions 2.1 [1-7] be satisfied, $\lambda > 2\omega$. Then, for all $p \ge 2$ and $T < +\infty$, the couple (u^*,y^*) is optimal at (t,x) - for the problem of minimizing (1.1)(2.2) - if and only if it is extremal.*

Theorem 2.13. *Let (u^*,y^*) be optimal at $(0,x)$ and let $\pi^*(\cdot;0,x)$ be the associated co-state. Then*

$$\Psi'(x) = \pi^*(0;0,x).$$

Consequently,

$$\Psi'(y^*(\tau)) = \pi^*(\tau;\tau,y^*(\tau)).$$

3. The motivating example

We here describe our motivating example: the infinite horizon problem of optimal investment with vintage capital, in the setting introduced by Barucci and Gozzi [12][13], and later reprised and generalized by Feichtinger et al. [32, 33, 34], and by Faggian [26, 27]. The capital accumulation is described by the following system

$$\begin{cases} \frac{\partial y(\tau,s)}{\partial \tau} + \frac{\partial y(\tau,s)}{\partial s} + \mu y(\tau,s) = u_1(\tau,s), & (\tau,s) \in]t,+\infty[\times]0,\bar{s}] \\ y(\tau,0) = u_0(\tau), & \tau \in]t,+\infty[\\ y(t,s) = x(s), & s \in [0,\bar{s}] \end{cases} \tag{3.1}$$

with $t > 0$ the initial time, $\bar{s} \in [0,+\infty]$ the maximal allowed age, and $\tau \in [0,T[$ with horizon $T = +\infty$. The unknown $y(\tau,s)$ represents the amount of capital goods of age s accumulated at time τ, the initial datum is a function $x \in L^2(0,\bar{s})$, $\mu > 0$ is a depreciation factor. Moreover, $u_0 : [t,+\infty[\to \mathbb{R}$ is the investment in new capital goods (u_0 is the boundary control) while $u_1 : [t,+\infty[\times[0,\bar{s}] \to \mathbb{R}$ is the investment at time τ in capital

goods of age s (hence, the distributed control). Investments are jointly referred to as the control $u = (u_0, u_1)$.
Besides, we consider the firm profits represented by the functional

$$I(t, x; u_0, u_1) = \int_t^{+\infty} e^{-\lambda\tau}[R(Q(\tau)) - c(u(\tau))]d\tau$$

where, for some given positive measurable coefficient α, we have that

$$Q(\tau) = \int_0^{\bar{s}} \alpha(s)y(\tau, s)ds$$

is the output rate (linear in $y(\tau)$) R is a concave revenue from $Q(\tau)$ (i.e., from $y(\tau)$). Moreover we have

$$c(u_0(\tau), u_1(\tau)) = \int_0^{\bar{s}} c_1(s, u_1(\tau, s))ds + c_0(u_0(\tau)),$$

with c_1 indicating the investment cost rate for technologies of age s, c_0 the investment cost in new technologies, including adjustment-innovation, c_0, c_1 convex in the control variables.
The entrepreneur's problem is that of maximizing $I(t, x; u_0, u_1)$ over all state–control pairs $\{y, (u_0, u_1)\}$ which are solutions in a suitable sense of equation (3.1). Such problems are known as *vintage capital* problems, for the capital goods depend jointly on time τ and on age s, which is equivalent to their dependence from time and vintage $\tau - s$.

When rephrased in an infinite dimensional setting, with $H := L^2(0, \bar{s})$ as state space, the state equation (3.1) can be reformulated as a linear control system with an unbounded control operator, that is

$$\begin{cases} y'(\tau) = A_0 y(\tau) + Bu(\tau), & \tau \in]t, +\infty[; \\ y(t) = x, \end{cases} \tag{3.2}$$

where $y : [t, +\infty[\to H$, $x \in H$, $A_0 : D(A_0) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{A_0 t}\}_{t\geq 0}$ on H with domain $D(A_0) = \{f \in H^1(0, \bar{s}) : f(0) = 0\}$ and defined as $A_0 f(s) = -f'(s) - \mu f(s)$, the control space is $U = \mathbb{R} \times H$, the control function is a couple $u \equiv (u_0, u_1) : [t, +\infty[\to \mathbb{R} \times H$, and the control operator is given by $Bu \equiv B(u_0, u_1) = u_1 + u_0\delta_0$, for all $(u_0, u_1) \in \mathbb{R} \times H$, δ_0 being the Dirac delta at the point 0. Note that, although $B \notin L(U, H)$, is $B \in L(U, D(A_0^*)')$. The reader can find in [12] the (simple) proof of the following theorem, which we will exploit in a short while.

Theorem 3.1. *Given any initial datum $x \in H$ and control $u \in L^p_\lambda(t, +\infty; U)$ the mild solution of the equation (3.2)*

$$y(s) = e^{(s-t)A}x + \int_t^s e^{(s-\tau)A}Bu(\tau)d\tau$$

belongs to $C([t, +\infty); H)$.

Following Remark 2.2, we then set

$$V = D(A_0^*) = \{f \in H^1(0, \bar{s}) : f(\bar{s}) = 0\}$$

and $V' = D(A_0^*)'$. Regarding the target functional, we set

$$J_\infty(t, x; u) := -I(t, x; u_0, u_1),$$

with:
$g_0 : V' \to \mathbb{R},\ g_0(x) = -R(\langle \alpha, x \rangle)$,
$h_0 : U \to \mathbb{R},\ h_0(u) = c_0(u_0) + \int_0^{\bar{s}} c_1(s, u_1(s))ds$.

Remark 3.2. Here the extension of the datum g_0 to V' is straightforward, as long as we assume that $\alpha \in V$ and replace scalar product in H with the duality in V, V'.
Note further that $\omega = 0$, $\lambda > 0$ (the type of the semigroup is negative and equal to $-\mu$). □

As the problem now fits into our abstract setting, the main results of the previous sections apply to the economic problem when data R, c_0, c_1 satisfy Assumption 2.1[8.a] or [8.b]. In particular, such thing happens in the following two interesting cases:

- If we assume, for instance, that R is a concave, C^1, sublinear function (for example one could take R quadratic in a bounded set and then take its linear continuation, see e.g. [32, 34]), and c_0, c_1 quadratic functions of the control variable, then Assumption 2.1[8.b] holds.
- Assumption 2.1[8.a] is instead satisfied when R is, for instance, quadratic - as it occurs in some other meaningful economic problems - and c_0, c_1 are equal to $+\infty$ outside some compact interval, and equal to any convex *l.s.c.* function otherwise. Such case corresponds to that of constrained controls (controls that violate the constrain yield infinite costs).

In these cases, Theorems 2.5, 2.6 hold true. In particular Theorem 2.6 states the existence of a unique optimal pair (u^*, y^*) for any initial datum $x \in V'$. Note that in general the optimal trajectory y^* lives in V'. However, since the economic problem makes sense in H, we would now like to infer that whenever x (the initial age distribution of capital) lies in H, then the whole optimal trajectory lives in H. Indeed, this is guaranteed by Theorem 3.1.
All these results allow to perform the analysis of the behavior of the optimal pairs and to study phenomena such as the diffusion of new technologies (see e.g. [12, 13]) and the anticipation effects (see e.g. [32, 34]). With respect to the results in [12, 13], here also the case of nonlinear R (which is particularly interesting from the economic point of view, as it takes into account the case of large investors) is considered. With respect to the results in [32, 34], here the existence of optimal feedbacks yields a tool to study more deeply the long run behavior of the trajectories, like the presence of long run equilibrium points and their properties.

4. Equilibrium points

We call *equilibrium point* any stationary solution of the closed loop equation

$$y'(\tau) = Ay(\tau) + B(h_0^*)'(-B^*\Psi'(y(\tau))),$$

that is any $x \in V'$ such that

$$Ax + B(h_0^*)'(-B^*\Psi'(x)) = 0. \tag{4.1}$$

Lemma 4.1. *Let Assumptions 2.1 [1-7] be satisfied, $p \geq 2$, $\lambda \geq 2\omega$. Any equilibrium point $x \in D(A)$ satisfies*

$$Ax + B(h_0^*)'(-B^*(\lambda - A_0^*)^{-1}g_0'(x)) = 0.$$

Proof. Let $\bar{x}$ be a solution to (4.1). Then there exist a stationary solution $\bar{p}$ to the co-state equation given by

$$\bar{p} := p^*(\tau;\bar{x}) = \int_\tau^\infty e^{(A_0^*-\lambda)(\sigma-\tau)}g_0'(\bar{x})d\sigma \equiv \int_0^\infty e^{(A_0^*-\lambda)r}g_0'(\bar{x})dr = (\lambda - A_0^*)^{-1}g_0'(\bar{x}),$$

where the last equality holds by definition of $(\lambda - A_0^*)^{-1}$. □

Whenever A proves invertible, then, equilibrium points can be regarded also as fixed points of the operator $T : V' \to V'$, defined by

$$Tx := -A^{-1}B(h_0^*)'(-B^*\Psi'(x)), \tag{4.2}$$

where Ψ is the unique strong solution of the HJB equation (2.4), or searched among solutions to

$$Tx := -A^{-1}B(h_0^*)'(-B^*(\lambda - A_0^*)^{-1}g_0'(x)), \tag{4.3}$$

Lemma 4.2. *Let Assumptions 2.1 [1-7] be satisfied and assume that*

$$\lambda - \omega > \|(A_0^*)^{-1}\|_{L(H)}\|B\|^2_{L(V',U)}[(h_0^*)'][g_0'].$$

Then there exists a unique solution $\bar{x} \in D(A)$ to (5.1).

To prove the assertion we need the following result.

Proposition 4.3. *Assume that $0 \in \rho(A_0^*)$ (that is, $(A_0^*)^{-1}$ is well defined and bounded in H). Then A^{-1} has bounded inverse on V', defined by the position*

$$\langle A^{-1}f, \varphi\rangle_{V,V'} = \langle f, (A_0^*)^{-1}\varphi\rangle_{V,V'}, \text{ for all } f \in V' \text{ and } \varphi \in V.$$

Moreover

$$\|A^{-1}\|_{L(V')} \leq \|(A_0^*)^{-1}\|_{L(H)}.$$

Proof. For all $f \in V'$ and $\varphi \in V$ we have

$$
\begin{aligned}
|\langle A^{-1} f, \varphi\rangle_{V,V'}| &= |\langle f, (A_0^*)^{-1}\varphi\rangle_{V,V'}| \\
&\le |f|_{V'} |(A_0^*)^{-1}\varphi|_V \\
&= |f|_{V'} (|(A_0^*)^{-1}\varphi|_H + |A_0^*(A_0^*)^{-1}\varphi|_H) \\
&= |f|_{V'} (|(A_0^*)^{-1}\varphi|_H + |(A_0^*)^{-1}A_0^*\varphi|_H) \\
&\le |f|_{V'} \|(A_0^*)^{-1}\|_{L(H)} |\varphi|_V
\end{aligned}
\tag{4.4}
$$

□

Proof. (Lemma 4.2) Define $T : V' \to V'$ as as in (4.2). Then T is Lipschitz continuous, with Lipschitz constant smaller than 1. Indeed, according to the content of the proof of Lemma 4.8 in [30] (with null final cost)

$$[\Psi'] \le \frac{[g_0']}{\lambda - \omega}$$

so that

$$|Tx - Ty|_{V'} \le \|(A_0^*)^{-1}\|_{L(H)} \|B\|^2_{L(V',U)} [(h_0^*)'] \frac{[g_0']}{\lambda - \omega} |x - y|_{V'}. \tag{4.5}$$

□

In the particular case of the economic problem described in Section 3 and with the data there defined, the preceding result reads as follows.

Corollary 4.4. *If either*

$$\lambda + \mu > \frac{1}{\mu}\ [(h_0^*)']\ [R']\ |\alpha|_V^2, \quad \textit{or} \quad \lambda + \mu > \frac{\bar{s}}{\sqrt{2}}\ [(h_0^*)']\ [R']\ |\alpha|_V^2,$$

then there exists a unique equilibrium point for optimal investment with vintage capital described in Section 3.

Proof. In the case of the economic problem we have $\|B\| \le 1$, $\omega = -\mu$. By means of Hille-Yosida Theorem

$$\|(A_0^*)^{-1}\|_{L(H)} \le \frac{1}{\mu}.$$

If we show that moreover

$$\|(A_0^*)^{-1}\|_{L(H)} \le \frac{\bar{s}}{\sqrt{2}},$$

the rest of the proof is straightforward. Recall that $D(A_0^*) = \{f \in H^1(0, \bar{s}) \ :\ f(\bar{s}) = 0\}$, moreover

$$(A_0^*)^{-1} f(s) = -\int_s^{\bar{s}} e^{-\mu(\sigma - s)} f(\sigma) d\sigma.$$

By means of Hölder inequality one derives

$$
\begin{aligned}
|(A_0^*)^{-1}f|_H^2 &= \int_0^{\bar{s}} \left| \int_s^{\bar{s}} e^{-\mu(\sigma - s)} f(\sigma) d\sigma \right|^2 ds \\
&\leq \frac{1}{2\mu} \int_0^{\bar{s}} (1 - e^{-2\mu(\bar{s}-s)}) |f|^2_{L^2(s,\bar{s})} ds \\
&\leq |f|_H^2 \frac{1}{2\mu} \left(\bar{s} - \frac{1 - e^{-2\mu\bar{s}}}{2\mu} \right) \\
&\leq |f|_H^2 \frac{\bar{s}^2}{2},
\end{aligned}
\tag{4.6}
$$

Instead if $\mu = 0$ we have

$$
|(A_0^*)^{-1}f|_H^2 \leq \int_0^{\bar{s}} \left(\frac{\bar{s}}{2} - \frac{(\bar{s} - \sigma)^2}{2} \right) |f(\sigma)|^2 d\sigma \leq \frac{\bar{s}^2}{2} |f|_H^2.
$$

□

Remark 4.5. If

$$
c(u) = \int_0^{\bar{s}} [\beta_1(s) u_1^2(s) + q_1(s) u_1(s)] ds + [\beta_0 u_0^2 + q_0 u_0], \quad R(Q) = bQ - Q^2,
$$

the preceding are implied respectively by

$$
\lambda + \mu > \frac{1}{\mu} \frac{a}{2} \left(1 + \frac{1}{\beta_0} \right) \left(\int_0^{\bar{s}} \alpha(s)^2 ds + \int_0^{\bar{s}} \alpha'(s)^2 ds - \mu\alpha(0)^2 \right)
$$

and

$$
\lambda + \mu > \frac{\bar{s}}{\sqrt{2}} \frac{a}{2} \left(1 + \frac{1}{\beta_0} \right) \left(\int_0^{\bar{s}} \alpha(s)^2 ds + \int_0^{\bar{s}} \alpha'(s)^2 ds - \mu\alpha(0)^2 \right).
$$

The statement derives from

$$
[g_0'] = 2a|\alpha|_V^2, \; [(h_0^*)'] = \| M_{\frac{1}{2\beta}} \|_{L(U)} \leq \frac{1}{2} \left(1 + \frac{1}{\beta_0} \right),
$$

where $M_{\frac{1}{2\beta}}$ is the operator described in (5.2) in the next section.

5. Explicit formulae for equilibrium points for vintage capital with convex-linear cost

Throughout the subsection we assume that h_0 is given by

$$
\begin{aligned}
h_0(u) &= (M_\beta u | u)_U + (q|u)_U \\
&= \int_0^{\bar{s}} u_1(s) [\beta_1(s) u_1(s) + q_1(s)] ds + u_0 [\beta_0 u_0 + q_0]
\end{aligned}
\tag{5.1}
$$

where $\beta = (\beta_0, \beta_1) \in R \times L^\infty(0, \bar{s})$, with $\beta_1(s), \beta_0 \geq \epsilon \geq 0$, $q = (q_0, q_1) \in R \times L^2(0, \bar{s}) \equiv U$, and $M_\beta : U \to U$ is given by

$$
M_\beta u(s) := (\beta_0 u_0, \beta_1(s) u(s)). \tag{5.2}
$$

Then is is easy to show that

$$
\begin{aligned}
h_0^*(u) &= (M_{\frac{1}{4\beta}}(u-q)|u-q)_U, \\
&= \int_0^{\bar{s}} \frac{1}{4\beta_1(s)}[u_1(s)-q_1(s)]^2 ds + \frac{1}{4\beta_0}[u_0-q_0]^2
\end{aligned}
\tag{5.3}
$$

so that

$$
(h_0^*)'(u) = M_{\frac{1}{2\beta}}(u-q),
$$

more explicitly

$$
(h_0^*)'(u)(s) = \left(\frac{1}{2\beta_0}[u_0-q_0]; \frac{1}{2\beta_1(s)}[u_1(s)-q_1(\cdot)]\right).
$$

Lemma 5.1. *Let* (5.1) *be satisfied, and* $R \in C^1(\mathbb{R})$, *with* R' *Lipschitz-continuous. Moreover, we set*

$$
w_1 = -A^{-1}BM_{\frac{1}{2\beta}}B^*(\lambda - A_0^*)^{-1}\alpha, \quad \textit{and} \quad w_2 = A^{-1}BM_{\frac{1}{2\beta}}q.
$$

Then there exists an equilibrium point $x \in H$ *if an only if there exists a real number* η *satisfying*

$$
\eta = R'(\langle \alpha, w_2 + \eta w_1 \rangle).
$$

In that case

$$
x = w_2 + \eta w_1.
$$

Moreover, if $R'' \leq 0$, *then such equilibrium point does exist and is unique.*

Remark 5.2. Note that w_1 and w_2 may be explicitly computed. According to the notation in [12], we set

$$
\bar{\alpha}(s) = (\lambda - A_0^*)^{-1}\alpha(s) = \int_s^{\bar{s}} e^{-(\mu+\lambda)(\sigma-s)}\alpha(\sigma)d\sigma
$$

the discounted return associated with a unit of capital of vintage s, and see that

$$
\begin{aligned}
w_1(s) &= -[A^{-1}BMB^*(\lambda - A_0^*)^{-1}\alpha](s) \\
&= -[A^{-1}BM(\bar{\alpha}(0), \bar{\alpha})](s) \\
&= -\left[A^{-1}B\left(\frac{\bar{\alpha}(0)}{2\beta_0}, \frac{\bar{\alpha}(\cdot)}{2\beta_1(\cdot)}\right)\right](s) \\
&= -\frac{\bar{\alpha}(0)}{2\beta_0}[A^{-1}\delta_0](s) - [A^{-1}\frac{\bar{\alpha}(\cdot)}{2\beta_1(\cdot)}](s) \\
&= \frac{\bar{\alpha}(0)}{2\beta_0}e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)}\frac{\bar{\alpha}(\sigma)}{2\beta_1(\sigma)}d\sigma.
\end{aligned}
\tag{5.4}
$$

Similarly, one shows that

$$
w_2(s) = \frac{q_0}{2\beta_0}e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)}\frac{q_1(\sigma)}{2\beta_1(\sigma)}d\sigma.
$$

Proof. In the case of the economic problem, $D(A) = H$, so that the operator $T : H \to H$ defined in (4.3) is given by

$$\begin{aligned} Tx &= -A^{-1}BM(-B^*\bar{p} - q) \\ &= -R'(\langle \alpha, x\rangle)A^{-1}BMB^*(\lambda - A_0^*)^{-1}\alpha + A^{-1}BMq \\ &= R'(\langle \alpha, x\rangle)w_1 + w_2. \end{aligned} \tag{5.5}$$

Hence

$$Tx = x \iff x - w_2 = R'(\langle \alpha, x\rangle)w_1$$

which is true if and only if $x - w_2 = \eta w_1$ for some $\eta \in \mathbb{R}$, that is if and only if

$$\eta = R'(\langle \alpha, w_2 + \eta w_1\rangle). \tag{5.6}$$

The last assertion is straightforward, as $\langle w_1, \alpha\rangle \geq 0$. □

From the preceding Lemma one derives the following results.

Lemma 5.3. *Let assumptions* (5.1) *be satisfied, and set* $c_1 := \langle \alpha, w_1\rangle$, $c_2 := \langle \alpha, w_2\rangle$. *Then there exists a unique equilibrium point* $\bar{x}$ *in each of the following cases:*

(i) If $R(Q) = -aQ^2 + bQ$, *then*

$$\bar{x} = w_2 - \frac{2ac_2 - b}{1 + 2ac_1} w_1;$$

(ii) If $R(Q) = \ln(1 + Q)$, *for* $Q \geq 0$ *and* $R(Q) = Q$ *for* $Q < 0$, *then*

$$\bar{x} = w_2 + \frac{\sqrt{(1 + c_2)^2 + 4c_1} - (1 + c_2)}{2c_1}\, w_1$$

(iii) If $R(Q) = (1 + Q)^\gamma - 1$, *with* $\gamma \in (0, 1)$, *for* $Q \geq 0$ *and* $R(Q) = \gamma Q$ *for* $Q < 0$, *then* $\bar{x} = w_1 + \bar{\eta} w_2$ *where* $\bar{\eta}$ *is the unique positive solution of*

$$\eta = \frac{\gamma}{(1 + c_1\eta + c_2)^{1-\gamma}}.$$

Proof. The proof is trivial. □

Remark 5.4. Results similar to those contained in Lemma 5.1, and 5.3 may be proved also in the case $h_0(u) = |u|_U^p$, $p \geq 2$.

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